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Continuous Hankel-Clifford Wavelet Transformation on Certain Distribution Spaces

V. R. Lakshmi Gorty

Department of Mathematics, SVKM's NMIMS University,
MPSTME, Vile-Parle (W), Mumbai -400056, India
lakshmi_gorty@rediffmail.com

Abstract

In this paper continuity of the Continuous Hankel-Clifford Wavelet Transform $H_{\nu,\psi}$ of function ϕ in terms of a mother wavelet ψ is investigated on certain distribution spaces when the Hankel transform of ψ defined by $\hat{\psi}(x,y) \in C^{\infty}\left(\mathrm{R}_{+}^{2}\right)$. A Sobolev space boundedness result is obtained.

***Mathematics Subject Classification**: 44A20, 42C40, 46, 33C05.*

***Keywords**: Continuous Hankel-Clifford wavelet, operator, Sobolev space, distribution spaces.*

1. Introduction

Méndez [8,9] investigated the following Hankel-Clifford transformation

$$F_1(y) = y^{\nu}\int_0^{\infty} (xy)^{-\nu/2} J_{\nu}\left[2\sqrt{xy}\right] f(x)\,dx \quad , \tag{1.1}$$

$J_{\nu}(x)$ being the Bessel function of the first kind of order $\nu \geq -1/2$. Throughout this paper it is assumed that $\nu \geq -1/2$ and $\phi \in L^1(R_+), R_+ = (0,\infty).$ In inversion formula, the function f can be recovered from its wavelet transform when the wavelet ψ satisfies admissibility condition as shown by Lakshmi Gorty in [3].

Theorem 1. Let $\psi \in L^2(\mathrm{R}_+)$ be a basic wavelet which defines continuous Hankel-Clifford wavelet transformation [3]. Then, for

$$A_{\psi} = \int_0^{\infty} w^{-2\nu-3/2}\left|\hat{\psi}(w)\right|^2 dw > 0, \tag{1.2}$$

which implies

$$\int_0^{\infty}\int_0^{\infty}\left(\left(H_{1,\nu,\psi} f\right)(b,a)\, f\right)(b,a)\overline{\left(\left(H_{1,\nu,\psi} f\right)(b,a)\, g\right)(b,a)}\; a^{-2\nu-3/2} da db = A_{\psi}\left\langle f,g\right\rangle \tag{1.3}$$

for all $f,g \in L^2(\mathrm{R}_+)$.

Using theory of H_{ν} space of Zemanian [2], Prasad [5] investigated the Hankel-Clifford wavelet transform B_{ψ} defined as follows:

$$\left(B_{\psi}\phi\right)(b,a)=\int_{0}^{\infty}(bu)^{\nu} J_{\nu}(bu)\hat{\phi}(u)\overline{\hat{\psi}(au)}\, du, \tag{1.4}$$

where $\hat{\phi}(u)=(h_{\nu}\phi)(u)$.

Assumed that for any real number $\rho,\hat{\psi}$ satisfies as in [6].

$$(1+x)^{l}\left|(xy)^{\frac{\nu-1}{2}}\hat{\psi}(xy)\right|\le C_{l,m,n}(1+y)^{\rho-n},\forall\ l,m,n\in N_{0}. \tag{1.5}$$

where $C_{l,m,n}>0$ is a constant and $\hat{\psi}$ denotes the Hankel-Clifford transform of the basic wavelet ψ. The class of all such wavelet $\hat{\psi}$ is denoted by $H_{1,\nu}^{\rho}$.

Thus the Hankel-Clifford transform with respect to x of $\hat{\psi}(ax)$,

$$h_{\nu}\left[\overline{\left(h_{\nu}(\psi)\right)}\right](a\xi)=\ \xi^{\nu}\int_{0}^{\infty}\ (x\xi)^{-\frac{\nu}{2}} J_{\nu}\left[2(x\xi)^{1/2}\right]\ \overline{\left(h_{\nu}(\psi)\right)}(ax)\,dx. \tag{1.6}$$

Notation and terminology of Méndez [9,10] is used. The differential operator $\Delta_{\nu}=x^{-\nu}D_{x}x^{\nu+1}D_{x}$ is defined by

$$\Delta_{\nu}=xD^{2}+(1+\nu)D \tag{1.7}$$

From [1, 2] it is noted that for any $\phi\in H_{\nu}$

$$h_{\nu}\left(\Delta_{\nu}\phi\right)=-yh_{\nu}\phi, \tag{1.8}$$

$$\left(d/dx\right)^{k}(\psi\phi)=\sum_{\omega=0}^{k}\binom{k}{\omega}\left(d/dx\right)^{\omega}\phi\left(d/dx\right)^{k-\omega}\psi \tag{1.9}$$

$$\Delta_{\nu}^{r}\phi(x)=\sum_{j=0}^{r}b_{j}x^{2j}\left(d/dx\right)^{r+j}\phi(x) \tag{1.10}$$

where b_{j} are constants depending only on ν.

Definition 1.1 A tempered distribution $\phi\in H_{\nu}'\left(R_{+}\right)$ is said to belong to the Sobolev space $G_{\nu}^{s,p}(\mathrm{R}_{+}), s,\nu\in\mathrm{R}, 1\le p<\infty,$ if its continuous Hankel-Clifford transform $h_{\nu}\phi$ corresponds to a locally integrable function over $\mathrm{R}_{+}=(0,\infty)$ such that

$$\|\phi\| G_{\nu}^{s,p}(\mathrm{R}_{+})=\left(\int_{0}^{\infty}\left|\ \left(1+\xi^{2}\right)^{s}(h_{\nu}\phi)(\xi)\right|^{p}\, d\xi\right)^{1/p}. \tag{1.11}$$

2. The Continuous Hankel-Clifford Wavelet Transform

Méndez [9] has defined the space $H_{2,\nu}$ and $H_{\nu,a}$ as follows:

Definition 2.1 Let ν be an arbitrary real number. $H_{2,\nu}$ denote the linear space consisting of all complex-valued smooth functions $\phi(x)$ on I such that for every pair of nonnegative integers (m,k), the number

$$\gamma_{m,k}^{2,\nu}(\phi)=\sup_{x\in R_+}\left|x^m(d/dx)^k\phi(x)\right|<\infty, k=0,1,2,... \tag{2.1}$$

Definition 2.2 From Méndez [9], if ν are arbitrary real parameters. Let a denote a positive real number. Then for each a and ν, we define $H_{\nu,a}$ as the space of testing functions $\phi(x)$ defined on $0<x<\infty$ and for which

$$\eta_k^{\nu,a}(\phi)=\sup_{0<x<\infty}\left|e^{-ax}\Delta_\nu^k\phi(x)\right|<\infty, k=0,1,2,... \tag{2.2}$$

The topology of the spaces $H_{2,\nu}$ and $H_{\nu,a}$ are generated by the seminorms $\left\{\gamma_{m,k}^{2,\nu}\right\}_{k=0}^{\infty}$ and $\left\{\eta_k^{\nu,a}\right\}_{k=0}^{\infty}$. It follows from Definition 2.1 and 2.2 that $H_{2,\nu}$ and $H_{\nu,a}$ are Fréchet spaces. We define

$$\sigma_{m,k}^{2,\nu}(\phi)=\max_{0\le\chi\le k}\gamma_{m,\chi}^{2,\nu}(\phi);\quad \rho_k^{\nu,a}(\phi)=\max_{0\le\chi\le k}\eta_\chi^{\nu,a}(\phi) \tag{2.3}$$

Then $\sigma_{m,k}^{2,\nu}$ and $\rho_k^{\nu,a}$ define a norm on the space $H_{2,\nu}$ and $H_{\nu,a}$ respectively. Following technique of Zemanian [2], we can write

$$x^{m+\nu}(d/dx)^n h_\nu\phi(x)=\int_0^\infty y^{\nu+1}\left(\frac{d^m}{dy^m}\phi(y)\right)\left\{x^{-\nu}J_{\nu+m}\left[2\sqrt{xy}\right]\right\}dy. \tag{2.4}$$

Theorem 2.3 The continuous Hankel-Clifford wavelet transform $B_{\nu,\psi}$ is a continuous linear mapping of $H_{2,\nu}$ into $H_{\nu,a}$.

Proof: Let $z=x+iy$ and $\nu\ge-1/2$, the continuous Hankel-Clifford wavelet transform $B_{\nu,\psi}$ has the representation $\left(B_{\nu,\psi}\phi\right)(z,a)=z\int_0^b (zu)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{zu}\right](h_\nu\phi)(u)\ \overline{\left(h_\nu\psi(au)\right)}du;\ \nu\ge-1/2.$ $b>0$ and $(h_\nu\phi)(u)\ \overline{\left(h_\nu\psi(au)\right)}\in L^2(0,b)$ if and only if $\left(B_{\nu,\psi}\phi\right)(z,a)\in L^2(0,\infty), z\left(B_{\nu,\psi}\phi\right)(z,a)$ is an even entire function of z and there exists a constant C such that in [7],

$\left|\left(B_{\nu,\psi}\phi\right)(z,a)\right|\le C\exp\left(b|y|\right),\forall z.$ Let $\phi\in H_{2,\nu}$, then

$$\left(B_{\nu,\psi}\phi\right)(z,a)=z^\nu\int_0^b (zu)^{-\frac{\nu}{2}}J_\mu\left[2\sqrt{zu}\right](h_\nu\phi)(u)\ \overline{\left(h_\nu\psi(au)\right)}du;\ \nu\ge-1/2.$$

$$=h_\nu\left[(h_\nu\phi)(u)\ \overline{\left(h_\nu\psi(au)\right)}\ \right](z).$$

Applying the technique of the Zemanian for fixed *a*, from (2.4),

$$\left(B_{\nu,\psi}\phi\right)(z,a)=\int_0^b\left[\frac{d^{2m}}{du^{2m}}\ \overline{\left(h_\nu\psi(au)\right)}(h_\nu\phi)(u)\right]\times\left\{u^{-\nu}J_{\nu+2m}\left[2\sqrt{zu}\right]\right\}du.$$

So that

$$\left|e^{-by^{2q}}\left(B_{\nu,\psi}\phi\right)(z,a)\right|$$

$$\leq \int_0^b \left|\left[(d/du)^{2m}\ \overline{\left(h_\nu\psi(au)\right)}\left(h_\nu\phi\right)(u)\right]\right| \times \left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right| du.$$

$$\leq \int_0^b \left|\sum_{s=0}^{2m}\binom{2m}{s}(d/du)^s\,\overline{\left(h_\nu\psi(au)\right)} \times (d/du)^{2m-s}\left(h_\nu\phi\right)(u)\right| \times \sup_{z,u}\left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right| du.$$

$$\leq \int_0^b \sum_{s=0}^{2m}\binom{2m}{s}\sup_u\left|(d/du)^s\,\overline{\hat{\psi}(au)}\ \right| \times \sup_u\left|(d/du)^{2m-s}\left(h_\nu\phi\right)(u)\right| \times \sup_{z,u}\left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right| du.$$

Applying in equalities (1.5) and (2.1), then from the above, we have

$$\int_0^b \sum_{s=0}^{2m}\binom{2m}{s}C_s(1+u)^\rho\,\gamma_{b,2m-s}^{2,\nu}\left(h_\nu\phi\right) \times \sup_{z,u}\left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right| du.$$

$$\leq \sum_{s=0}^{2m}\binom{2m}{s}C_s(1+u)^\rho\,\gamma_{b,2m-s}^{2,\nu}\left(h_\nu\phi\right)\sup_{z,u}\left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right|\int_0^b du. \tag{2.5}$$

$$\leq \sum_{s=0}^{2m}\binom{2m}{s}C_s(1+u)^\rho\,\gamma_{b,2m-s}^{2,\nu}\left(h_\nu\phi\right)\left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right|.$$

$$\left|e^{-by^{2q}}\left(B_{\nu,\psi}\phi\right)(z,a)\right| \leq \sum_{s=0}^{2m}\binom{2m}{s}C_s(1+u)^\rho\,\gamma_{b,2m-s}^{2,\nu}\left(h_\nu\phi\right)\left|J_{\nu+2m}\left[2\sqrt{zu}\right]e^{-by^{2q}}\right|.$$

This completes the proof of the theorem.

3. The Sobolev Type Space

The Sobolev space $G_\nu^{s,p}(\mathrm{R}_+)$ is defined as in [5] by (1.11). In the following, we shall make use of the following norm on $G_\nu^{s,p}(\mathrm{R}_+\times\mathrm{R}_+)$ in the proof of the boundedness result

$$\|\phi\|G_\nu^{s,p}(\mathrm{R}_+) = \left(\int_0^\infty\int_0^\infty |\ \left(1+\xi^2\right)^s\left(1+\eta^2\right)^s\,\overline{\left(h_\nu\phi\right)(\xi,\eta)}|^p\,d\xi d\eta\right)^{1/p},\ \phi\in H_\nu'(\mathrm{R}_+\times\mathrm{R}_+).$$

Lemma 3.1 Assume that for any positive real number ρ, $\hat{\psi}(x)$ satisfies

$$\left|(d/dx)^l\,\hat{\psi}(x)\right| \leq C_{l,\rho}(1+x)^{\rho-l},\ \forall\ l\in N_0. \tag{3.1}$$

then there exists a positive constant C' *such that*

$$\left|h_\nu\left[\overline{\left(h_\nu(\psi)\right)}\right](a\xi)\right| \leq C'(1+a)^{\rho+2l}\left(1+\xi^2\right)^{-l}. \tag{3.2}$$

Proof: From the definition (1.6), it implies

$$h_\nu\left[\overline{(h_\nu\psi)}\right](a\xi) = \xi^\nu \int_0^\infty (x\xi)^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{x\xi}\right] \overline{(h_\nu\psi)}(ax)\,dx.$$

So that from [4],

$$\left(1+\xi^2\right)^l h_\nu\left[\overline{(h_\nu\psi)}\right](a\xi) = \xi^\nu \int_0^\infty (x\xi)^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{x\xi}\right]\left(1-\Delta_{\nu,x}\right)^l \overline{(h_\nu\psi)}(ax)\,dx. \quad (3.3)$$

$\forall\ \xi, a \in R_+, l \in N_0$ and $\Delta_{\nu,x}$ defined as (1.7). Now,

$$\begin{aligned}\left(1-\Delta_{\nu,x}\right)^l \overline{(h_\nu\psi)}(ax) &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r \Delta_{\nu,x}^r \overline{(h_\nu\psi)}(ax)\\ &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r \sum_{j=0}^{r} b_j x^{2j}\left(d/dx\right)^{r+j}\left(\overline{(h_\nu\psi)}(ax)\right) \quad (3.4)\end{aligned}$$

Hence by (3.3) and (3.4), and inequality (3.1), we have

$$\begin{aligned}&\left|h_\nu\left[\overline{\left(h_\nu(\psi)\right)}\right](a\xi)\right|\\ &= \left|\left(1+\xi^2\right)^{-l} \xi^\nu \sum_{r=0}^{l}\binom{l}{r}(-1)^r \int_0^\infty\left[(x\xi)^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{x\xi}\right]\sum_{j=0}^{r} b_j x^{2j}\left(d/dx\right)^{r+j}\left(\overline{(h_\nu\psi)}(ax)\right)\right]dx\right|\\ &\le \left|\left(1+\xi^2\right)^{-l} \xi^\nu \sum_{r=0}^{l}\sum_{j=0}^{r} b_j \binom{l}{r}(-1)^r \int_0^\infty\left[(x\xi)^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{x\xi}\right] x^{2j}\left(d/dx\right)^{r+j}\hat{\psi}(ax)\right]dx\right|\\ &\le (1+a)^{\rho+2l}\left(1+\xi^2\right)^{-l}\xi^\nu \sum_{r=0}^{l}\sum_{j=0}^{r}(-1)^r b_j\binom{l}{r} C_{r+j,\rho}\int_0^\infty (1+x)^{2j+\rho-2l}(x\xi)^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{x\xi}\right]dx.\end{aligned}$$

choosing $l - j > \rho/2$, conclude that

$$\left|h_\nu\left[\overline{\left(h_\nu(\psi)\right)}\right](a\xi)\right| \le C'(1+a)^{\rho+2l}\left(1+\xi^2\right)^{-l}.$$

Definition 3.2 Let $(h_\nu\psi)(a\xi)$ be a wavelet in H_ν^ρ defined by (1.5). Then the continuous Hankel-Clifford wavelet transform $B_{\nu,\psi}$ has the representation

$$\left(B_{\nu,\psi}\phi\right)(y,x) = y^\nu \int_0^\infty (y\eta)^{-\frac{\nu}{2}} J_\nu\left[2\sqrt{y\eta}\right]\overline{\left(h_\nu\psi(x\eta)\right)}\left(h_\nu\phi\right)(\eta)\,d\eta \text{ exists for } \phi\in H_{2,\nu}\left(R_+\right). \quad (3.5)$$

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Theorem 3.3 *For any wavelet* $h_\nu\psi \in H_\nu^\rho$ *the continuous Hankel-Clifford wavelet transform* $B_{\nu,\psi}$ *admits the representation*

$$\left(B_{\nu,\psi}\phi\right)=\int_0^\infty\int_0^\infty\left[(xy)^\nu(x\xi)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{x\xi}\right](y\eta)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{y\eta}\right]\right]\overline{(h_\nu\hat{\psi})(\xi\eta)}(h_\nu\phi)(\eta)\,d\xi d\eta. \qquad (3.6)$$

exists for $\phi \in H_{2,\nu}(R_+)$.

Proof: From definition (3.5)

$$\left(B_{\nu,\psi}\phi\right)(y,x)$$

$$=y^\nu\int_0^\infty (y\eta)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{y\eta}\right]\overline{\left(h_\nu\psi(x\eta)\right)}(h_\nu\phi)(\eta)\,d\eta$$

$$=y^\nu\int_0^\infty\left[(y\eta)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{y\eta}\right]\left[x^\nu\int_0^\infty (x\xi)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{x\xi}\right]\overline{(h_\nu\hat{\psi})(\xi\eta)}\,d\xi\right](h_\nu\phi)(\eta)\right]d\eta$$

$$=\int_0^\infty\int_0^\infty (xy)^\nu(x\xi)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{x\xi}\right](y\eta)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{y\eta}\right](h_\nu\hat{\psi})(\xi\eta)(h_\nu\phi)(\eta)\,d\xi d\eta.$$

The last integral exists because $(h_\nu\phi)(\eta)\in H_{2,\nu}(R_+)$ and $\overline{(h_\nu\hat{\psi})(\xi\eta)}$ satisfies the inequality (3.2).

Corollary 3.4 *For any wavelet* $h_\nu\psi \in H_\nu^\rho$ *the continuous Hankel-Clifford wavelet transform* $h_\nu\left(B_{\nu,\psi}\phi\right)(\xi,\eta)$ *admits the representation*

$$h_\nu\left(B_{\nu,\psi}\phi\right)(\xi,\eta)=\overline{\left(h_\nu\hat{\psi}(\xi\eta)\right)}\left(h_\nu\hat{\phi}\right)(\eta), \qquad (3.7)$$

where $\phi \in H_{2,\nu}(R_+)$.

Proof: The right hand side of (3.7)

$$\overline{\left(h_\nu\hat{\psi}(\xi\eta)\right)}\left(h_\nu\hat{\phi}\right)(\eta)$$

$$=h_\nu\left[\overline{(h_\nu\psi)(\xi\eta)}\right]h_\nu\left[(h_\nu\phi)(\eta)\right]$$

$$=\xi^\nu\int_0^\infty (x\xi)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{x\xi}\right]\overline{(h_\nu\psi)(x\eta)}\,dx\times\eta^\nu\int_0^\infty (y\eta)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{y\eta}\right](h_\nu\phi)(y)\,dy$$

$$=(\xi\eta)^\nu\int_0^\infty\int_0^\infty\left[(x\xi)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{x\xi}\right](y\eta)^{-\frac{\nu}{2}}J_\nu\left[2\sqrt{y\eta}\right]\overline{(h_\nu\psi)(x\eta)}(h_\nu\phi)(y)\right]dxdy$$

$$=h_\nu\left(B_{\nu,\psi}\phi\right)(\xi,\eta).$$

Theorem 3.5 *Let* $h_\nu\psi \in H_\nu^\rho$ *and* $\left(B_{\nu,\psi}\phi\right)(y,x)$ *be the continuous Hankel-Clifford wavelet transform then there exists* $D>0$ *such that for* $\rho \in R_+$ and $l \in N_0$,

$$\left\|\left(B_{\nu,\psi}\phi\right)\right\|G_\nu^{s,p}(R_+\times R_+)\le D\left\|(h_\nu\phi)\right\|G_\nu^{s+(\rho+2l)/2,p}(R_+),\forall\phi\in H_{2,\nu}(R_+).$$

Proof: Using Lemma 3.1, we have

$$\left\|\left(B_{\nu,\psi}\phi\right)\right\|G_{\nu}^{s,p}\left(R_{+}\times R_{+}\right)$$

$$=\left(\int_0^\infty\int_0^\infty \left|\left(1+\xi^2\right)^s\left(1+\eta^2\right)^s h_\nu\left(B_{\nu,\psi}\phi\right)(\xi,\eta)\right|^p d\xi d\eta\right)^{1/p}$$

$$=\left(\int_0^\infty\int_0^\infty \left|\left(1+\xi^2\right)^s\left(1+\eta^2\right)^s \overline{\left(h_\nu\hat{\psi}(\xi\eta)\right)}\left(h_\nu\hat{\phi}\right)(\eta)\right|^p d\xi d\eta\right)^{1/p}$$

$$\leq\left(\int_0^\infty\int_0^\infty \left|\left(1+\xi^2\right)^s\left(1+\eta^2\right)^s\right|^p \left|h_\nu\overline{\left(h_\nu\psi\right)(\xi\eta)}\right|^p\left|\left(h_\nu\hat{\phi}\right)(\eta)\right|^p d\xi d\eta\right)^{1/p}$$

$$\leq\left(\int_0^\infty\int_0^\infty \left|\left(1+\xi^2\right)^s\left(1+\eta^2\right)^s C'(1+\eta)^{\rho+2l}\left(1+\xi^2\right)^{-l}\left(h_\nu\hat{\phi}\right)(\eta)\right|^p d\xi d\eta\right)^{1/p}.$$

Note that

$$(1+\eta)^{\rho+2l}\leq 2^{(\rho+2l)/2}\left(1+\eta^2\right)^{(\rho+2l)/2},\rho\geq 0$$

and

$$(1+\eta)^{\rho+2l}\leq\left(1+\eta^2\right)^{(\rho+2l)/2},\rho<0.$$

Therefore

$$(1+\eta)^{\rho+2l}\leq\max\left(1,2^{(\rho+2l)/2}\right)\left(1+\eta^2\right)^{(\rho+2l)/2}.$$

Hence

$$\left\|\left(B_{\nu,\psi}\phi\right)\right\|G_{\nu}^{s,p}\left(R_{+}\times R_{+}\right)$$

$$\leq\left(\int_0^\infty\int_0^\infty \left|\left(1+\xi^2\right)^{s-l}\left(1+\eta^2\right)^s C'\max\left(1,2^{(\rho+2l)/2}\right)\left(1+\eta^2\right)^{(\rho+2l)/2}\left(h_\nu\hat{\phi}\right)(\eta)\right|^p d\xi d\eta\right)^{1/p}.$$

$$\leq C''\left(\int_0^\infty\left|\left(1+\eta^2\right)^{s+(\rho+2l)/2}\left(h_\nu\hat{\phi}\right)(\eta)\right|^p d\eta\right)^{1/p}\left(\int_0^\infty\left|\left(1+\xi^2\right)^{s-l}\right|^p d\xi\right)^{1/p}.$$

where C'' is certain constant. The ξ integral is convergent as l can be chosen large enough so that $\left\|\left(B_{\nu,\psi}\phi\right)\right\|G_{\nu}^{s,p}\left(R_{+}\times R_{+}\right)\leq D\left\|\left(h_\nu\phi\right)\right\|G_{\nu}^{s+(\rho+2l)/2,p}\left(R_{+}\right).$

4. Product of two continuous Hankel-Clifford wavelet transforms

Let B_{ν,ψ_1} and B_{ν,ψ_2} be two continuous Hankel-Clifford wavelet transforms of $\forall\phi\in H_{2,\nu}\left(R_{+}\right)$ defined as follows:

$$\left(B_{\nu,\psi_1}\phi\right)(b,a) = B_1(b,a)$$
$$= b^{\nu}\int_0^{\infty} (bu)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{bu}\right]\overline{\left(h_{\nu}\psi_1(au)\right)}\left(h_{\nu}\phi\right)(u)\,du$$

and

$$\left(B_{\nu,\psi_2}\phi\right)(d,c) = B_2(d,c)$$
$$= d^{\nu}\int_0^{\infty} (du)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{du}\right]\overline{\left(h_{\nu}\psi_2(cu)\right)}\left(h_{\nu}\phi\right)(u)\,du$$

which can be written as

$$\left(B_{\nu,\psi_1}\phi\right)(b,a) = b^{\nu}\int_0^{\infty} (bu)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{bu}\right]\overline{\hat{\psi}_1(au)}\,\hat{\phi}(u)\,du \tag{4.1}$$

and

$$\left(B_{\nu,\psi_2}\phi\right)(d,c) = d^{\nu}\int_0^{\infty} (du)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{du}\right]\overline{\hat{\psi}_2(cu)}\,\hat{\phi}(u)\,du \tag{4.2}$$

Then, their product $B_{\nu,\psi_1} o\, B_{\nu,\psi_2}$ or $B_1 o B_2$ is defined by

$$B(b,a,c) = \left(B_1 o B_2\right)(b,a,c)$$
$$= b^{\nu}\int_0^{\infty} (bu)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{bu}\right]\overline{\hat{\psi}_1(au)}\left[h_{\nu}\left(B_{\nu,\psi_2}\phi\right)\right](u,c)\,du \tag{4.3}$$
$$= b^{\nu}\int_0^{\infty} (bu)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{bu}\right]\overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(cu)}\,\hat{\phi}(u)\,du \tag{4.4}$$
$$= b^{\nu}\int_0^{\infty} (bu)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{bu}\right]\chi(a,c,u)\,\hat{\phi}(u)\,du.$$

where $\hat{\phi}$ denotes the continuous Hankel-Clifford wavelet transformation of ϕ and $\chi(a,c,u) = \overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(cu)}$, provided the integral is convergent.

Theorem 4.1 Let $\overline{\hat{\psi}_1(au)} \in H_{\nu}^{\rho_1}$ *and* $\overline{\hat{\psi}_2(cu)} \in H_{\nu}^{\rho_2}$ *, then for certain constant exists* $C_2 > 0$ *such that for* $\rho_1, \rho_2 \in R_+$,

$$\left\|\left(B_{\nu,\psi_1}B_{\nu,\psi_2}\phi\right)(b,a,c)\right\| G_{\nu}^{s,p}\left(R_+ \times R_+\right) \le C_2 \left\|\hat{\phi}\right\| G_{\nu}^{s+2l+(\rho_1+\rho_2)/2,\,p}\left(R_+\right).$$

Proof: By definition (4.3),

$$\left(B_{\nu,\psi_1}B_{\nu,\psi_2}\phi\right)(b,a,c) = b^{\nu}\int_0^{\infty} (bu)^{-\frac{\nu}{2}} J_{\nu}\left[2\sqrt{bu}\right]\overline{\hat{\psi}_1(au)}\left[h_{\nu}\left(B_{\nu,\psi_2}\phi\right)\right](u,c)\,du.$$

From (4.4), it follows that $\left(B_{\nu,\psi_1}B_{\nu,\psi_2}\phi\right)$ has continuous Hankel-Clifford wavelet transformation equal to $\overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(cu)}\,\hat{\phi}(u)$.

Therefore

$$\left\|\left(B_{\nu,\psi_1}B_{\nu,\psi_2}\phi\right)(b,a,c)\right\|G_\nu^{s,p}\left(R_+\times R_+\right)$$

$$=\left(\int_0^\infty\int_0^\infty\left|\ \left(1+u^2\right)^s\left(1+a^2\right)^s\overline{\hat{\psi}_1(au)}\,\overline{\hat{\psi}_2(au)}\,\hat{\phi}(u)\right|^p dadu\right)^{1/p}.$$

Since from (1.5)

$$\left|\overline{\hat{\psi}_1(au)}\right|\le C_{\rho_1,l}\left(1+u\right)^{\rho_1+2l}\left(1+a\right)^{-l}$$

$$\le C_{\rho_1,l}\max\left(1,2^{(\rho_1+2l)/2}\right)\left(1+u^2\right)^{(\rho_1+2l)/2}\max\left(1,2^{-l/2}\right)\left(1+a^2\right)^{-l/2}.$$

and

$$\left|\overline{\hat{\psi}_2(au)}\right|\le C_{\rho_2,l}\left(1+u\right)^{\rho_2+2l}\left(1+a\right)^{-l}$$

$$\le C_{\rho_2,l}\max\left(1,2^{(\rho_2+2l)/2}\right)\left(1+u^2\right)^{(\rho_2+2l)/2}\max\left(1,2^{-l/2}\right)\left(1+a^2\right)^{-l/2}.$$

Therefore

$$\left\|\left(B_{\nu,\psi_1}B_{\nu,\psi_2}\phi\right)(b,a,c)\right\|G_\nu^{s,p}\left(R_+\times R_+\right)$$

$$\le\left(\begin{array}{l}\int_0^\infty\int_0^\infty\left|\ \left(1+u^2\right)^s\left(1+a^2\right)^s\right.\\ \times C_{\rho_1,l}\max\left(1,2^{(\rho_1+2l)/2}\right)\left(1+u^2\right)^{(\rho_1+2l)/2}\max\left(1,2^{-l/2}\right)\left(1+a^2\right)^{-l/2}\\ \times C_{\rho_2,l}\max\left(1,2^{(\rho_2+2l)/2}\right)\left(1+u^2\right)^{(\rho_2+2l)/2}\max\left(1,2^{-l/2}\right)\left(1+a^2\right)^{-l/2}\ \hat{\phi}(u)\left.\right|^p dadu\end{array}\right)^{1/p}.$$

$$\le\ C_{\rho_1,\rho_2,l}\left(\int_0^\infty\left|\left(1+a^2\right)^{s-l}\right|^p da\right)^{1/p}\left(\int_0^\infty\left|\ \left(1+u^2\right)^{s+2l+(\rho_1+\rho_2)/2}\hat{\phi}(u)\right|^p du\right)^{1/p}$$

Where $C_{\rho_1,\rho_2,l}$ is certain positive constant. The right hand side of the integral can be made convergent by choosing l sufficiently large, so that

$$\left\|\left(B_{\nu,\psi_1}B_{\nu,\psi_2}\phi\right)(b,a,c)\right\|G_\nu^{s,p}\left(R_+\times R_+\right)\le C_2\left\|\hat{\phi}\right\|G_\nu^{s+2l+(\rho_1+\rho_2)/2,p}\left(R_+\right),$$

Where C_2 is positive constant.

References

[1] R. S. Pathak, Integral Transforms of Generalized Functions and Their Applications, Gordon and Breach Publishers, Amsterdam. (1997).
[2] A.H. Zemanian Generalized Integral Transformations, Interscience Publishers, New York (1968).
[3] V. R. Lakshmi Gorty, Continuous Hankel-Clifford wavelet transformation, Journal of Wavelet Theory and Applications, (JWTA), ISSN 0973-6336 Volume 7, Number 1, pp. 45-55, (2013).
[4] L. Debnath and D. Bhatta, Integral transforms and their applications, second. ed. Chapman and Hall/CRC Press, Boca Raton, FL, (2007).
[5] Akhilesh Prasad, Asutosh Mahato and M. M. Dixit, The Bessel Wavelet Transform, Int. Journal of Math. Analysis, vol. 5, no. 2, 87-97, (2011).
[6] G. N. Watson, A treatise on the theory of Bessel functions, Cambridge University Press, London, (1958).
[7] R. S. Pathak, The Wavelet Transform, Atlantis Studies in Mathematics for Engineering and Science: Atlantis Press, Vol. 4, (2009).
[8] J. M. R. Méndez-Pérez and M. Socas Robayana, La Transformacion Integral De Hankel-Clifford De Orden Arbitrario Homenajeal Prof. Dr. Nacere Hayek Calil pp.199-207, (1990).
[9] J. M. Méndez, La transformación integral de Hankel-Clifford, Tesis, Secretariado de Publicaciones, Univ. La Laguna, Col. Monog. 8, (1981).
[10] K. Trim`eche,Generalized Wavelets and Hypergroups, Gordon and Breach Science Publishers, Amsterdam (1997).